THE ASYMPTOTIC FOR THE SECOND MOMENT OF $\zeta(s)$ ON THE CRITICAL LINE

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Abstract. In 1918 Hardy and Littlewood [6] showed that

$$I_1(T) := \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt \sim T\log T$$

for sufficiently large $T > 0$, where $\zeta(s)$ is the Riemann zeta-function. This short note is devoted to presenting a proof of this result using the technique of approximating $\zeta(s)$ by special Dirichlet polynomials. Our exposition is largely based on [14] with some adaptations. We shall also describe briefly Titchmarsh's proof of this asymptotic and discuss higher moments of $\zeta(s)$ on the critical line.

1. The Euler-Maclaurin Summation Formula

Let $f\colon [M,N] \to \mathbb{C}$ be a continuously differentiable function on $[M,N]$, where $N \ge M \ge 0$ are integers. The Euler-Maclaurin formula [14, Equ (2.1)] states

$$\sum_{M<n\le N} f(x) = \int_M^N (f(x)+\psi(x)f'(x))\,dx + \frac{f(N)-f(M)}{2},$$

where $\psi(x) = x - \lfloor x\rfloor - 1/2 = \{x\} - 1/2$. Here $\lfloor x\rfloor$ denotes the integer part of x and $\{x\} = x - \lfloor x\rfloor$ the fractional part of x. The Euler-Maclaurin formula can be derived easily by integration by parts.

Suppose now that $[a,b] \subseteq [M,N]$ is a subinterval, where $a,b \in \mathbb{R}$. Then we have

$$\begin{aligned}\sum_{M<n\le b} f(x) &= \int_M^{\lfloor b\rfloor} (f(x)+\psi(x)f'(x))\,dx + \frac{f(\lfloor b\rfloor)-f(M)}{2}\\ &= \int_M^{\lfloor b\rfloor} (f(x)+\{x\}f'(x))\,dx.\end{aligned}$$

Note that

$$\int_{\lfloor b\rfloor}^b \{x\}f'(x)\,dx = \int_{\lfloor b\rfloor}^b (x-\lfloor b\rfloor)f'(x)\,dx = \{b\}f(b) - \int_{\lfloor b\rfloor}^b f(x)\,dx.$$

Hence

$$\begin{aligned}\sum_{M<n\le b} f(x) &= \int_M^b (f(x)+\{x\}f'(x))\,dx - \{b\}f(b)\\ &= \int_M^b (f(x)+\psi(x)f'(x))\,dx + \frac{f(b)-f(M)}{2} - \{b\}f(b).\end{aligned}$$

Similarly, we have

$$\sum_{M<n\le a} f(x) = \int_M^a (f(x)+\psi(x)f'(x))\,dx + \frac{f(a)-f(M)}{2} - \{a\}f(a).$$

It follows that

$$\begin{aligned}\sum_{a<n\le b} f(x) &= \int_a^b (f(x)+\psi(x)f'(x))\,dx + \frac{f(b)-f(a)}{2} - (\{b\}f(b)-\{a\}f(a)) \\ &= \int_a^b (f(x)+\psi(x)f'(x))\,dx - (\psi(b)f(b)-\psi(a)f(a)). \qquad (1)\end{aligned}$$

We shall apply this formula in the next section to estimate the sum $\sum_{a<n\le b} n^{-s}$.

2. Technical Lemmas

In this section we prove two technical results needed for estimating the sum $\sum_{a<n\le b} n^{-s}$. Such results are useful in estimating exponential sums of certain types. The author learned these results from [14, Chapter 2].

Lemma 2.1. *Let $g\colon [a,b]\to\mathbb{R}$ be a twice continuously differentiable function on $[a,b]$ such that $g'(x)g''(x)\neq 0$ for all $x\in[a,b]$. If $h\colon [a,b]\to\mathbb{C}$ is any continuously differentiable function on $[a,b]$, then*

$$\left|\int_a^b h(x)e^{2\pi i g(x)}\,dx\right| \le \frac{H}{\pi}\left(\frac{1}{|g'(a)|}+\frac{1}{|g'(b)|}\right),$$

where

$$H = \max(|h(a)|,|h(b)|) + \int_a^b |h'(x)|\,dx. \qquad (2)$$

Proof. Since $g''(x)$ never vanishes on $[a,b]$, it follows that $g'(x)$ is monotone on $[a,b]$. But $g'(x)\neq 0$ for all $x\in[a,b]$. Thus $1/g'(x)$ and $|g'(x)|$ are both monotone on $[a,b]$. Note that

$$2\pi i\int_a^b e^{2\pi i g(x)}\,dx = \frac{e^{2\pi i g(b)}}{g'(b)} - \frac{e^{2\pi i g(a)}}{g'(a)} - \int_a^b e^{2\pi i g(x)}\,d\left(\frac{1}{g'(x)}\right)$$

with

$$\left|\int_a^b e^{2\pi i g(x)}\,d\left(\frac{1}{g'(x)}\right)\right| \le \left|\int_a^b d\left(\frac{1}{g'(x)}\right)\right| = \left|\frac{1}{g'(b)}-\frac{1}{g'(a)}\right| \le \frac{1}{|g'(a)|}+\frac{1}{|g'(b)|}.$$

Hence

$$2\pi\left|\int_a^b e^{2\pi i g(x)}\,dx\right| \le 2\left(\frac{1}{|g'(a)|}+\frac{1}{|g'(b)|}\right),$$

which gives

$$\left|\int_a^b e^{2\pi i g(x)}\,dx\right| \le \frac{1}{\pi}\left(\frac{1}{|g'(a)|}+\frac{1}{|g'(b)|}\right). \qquad (3)$$

Suppose first that $|g'(x)|$ is decreasing on $[a,b]$. By integration by parts we have

$$\int_a^b h(x)e^{2\pi i g(x)}\,dx = h(b)\int_a^b e^{2\pi i g(x)}\,dx - \int_a^b\left(\int_a^x e^{2\pi i g(y)}\,dy\right)h'(x)\,dx.$$

It follows by (3) that

$$\begin{aligned}\left|\int_a^b h(x)e^{2\pi i g(x)}\,dx\right| &\le |h(b)|\left|\int_a^b e^{2\pi i g(x)}\,dx\right| + \int_a^b \left|\int_a^x e^{2\pi i g(y)}\,dy\right| |h'(x)|\,dx\\ &\le \frac{|h(b)|}{\pi}\left(\frac{1}{|g'(a)|}+\frac{1}{|g'(b)|}\right) + \int_a^b \frac{1}{\pi}\left(\frac{1}{|g'(a)|}+\frac{1}{|g'(x)|}\right)|h'(x)|\,dx\\ &\le \frac{1}{\pi}\left(|h(b)| + \int_a^b |h'(x)|\,dx\right)\left(\frac{1}{|g'(a)|}+\frac{1}{|g'(b)|}\right).\end{aligned}$$

If $|g'(x)|$ is increasing on $[a,b]$, we have

$$\int_a^b h(x)e^{2\pi i g(x)}\,dx = h(a)\int_a^b e^{2\pi i g(x)}\,dx + \int_a^b \left(\int_x^b e^{2\pi i g(y)}\,dy\right) h'(x)\,dx.$$

By the same argument we obtain

$$\left|\int_a^b h(x)e^{2\pi i g(x)}\,dx\right| \le \frac{1}{\pi}\left(|h(a)| + \int_a^b |h'(x)|\,dx\right)\left(\frac{1}{|g'(a)|}+\frac{1}{|g'(b)|}\right).$$

This completes the proof of the lemma. □

Lemma 2.2. *Let $\theta \in [0,1)$ and let $g\colon [a,b] \to \mathbb{R}$ be a twice continuously differentiable function on $[a,b]$ such that $|g'(x)| \le \theta$ and $g''(x) \ne 0$ for all $x \in [a,b]$. If $h\colon [a,b] \to \mathbb{C}$ is any continuously differentiable function on $[a,b]$, then*

$$\left|\int_a^b h(x)\psi(x)e^{2\pi i g(x)}\,dx\right| \le \frac{4H}{\pi^2(1-\theta)},$$

where H is defined as in (2).

Proof. Let $n \in \mathbb{Z}\setminus\{0\}$ be a nonzero integer. Applying Lemma 2.1 with $g(x)$ replaced by $g(x)+nx$ yields

$$\left|\int_a^b h(x)e^{2\pi i(g(x)+nx)}\,dx\right| \le \frac{2H}{\pi(|n|-\theta)}. \tag{4}$$

The function $\psi(x)$ is piecewise linear on $\mathbb{R}$, periodic of period 1, and smooth on $(n,n+1)$ for every $n \in \mathbb{Z}$ with jump discontinuities at integer points. Its Fourier expansion is

$$\psi(x) = \sum_{0<|n|\le N} \frac{e^{2\pi i n x}}{2\pi i n} + O\left(\frac{1}{1+\|x\|N}\right),$$

where $N \ge 1$ and $\|x\|$ is the shortest distance of x to $\mathbb{Z}$. It follows by (4) that

$$\left|\int_a^b h(x)\psi(x)e^{2\pi i g(x)}\,dx\right| \le \sum_{n=1}^{\infty} \frac{2H}{\pi^2 n(n-\theta)} + O\left(\int_a^b \frac{|h(x)|}{1+\|x\|N}\,dx\right).$$

Since

$$\sum_{n=1}^{\infty} \frac{2H}{\pi^2 n(n-\theta)} \le \frac{2H}{\pi^2(1-\theta)}\left(1+\sum_{n=2}^{\infty}\frac{1}{n(n-1)}\right) = \frac{4H}{\pi^2(1-\theta)},$$

we have

$$\left|\int_a^b h(x)\psi(x)e^{2\pi i g(x)}\,dx\right| \le \frac{4H}{\pi^2(1-\theta)} + O\left(\int_a^b \frac{|h(x)|}{1+\|x\|N}\,dx\right).$$

We finish the proof of the lemma by letting $N \to \infty$. $\square$

Let $N \geq 1$ be a positive integer and let $T \in \mathbb{R}$ with $1 \leq T \leq N$. Let $s = \sigma + it \in \mathbb{C}$ with $\sigma > 0$ and $|t| \leq 2T$. Applying (1) with $f(x) = h(x)e^{2\pi i g(x)}$, where $h(x) = x^{-\sigma}$ and $g(x) = -(t/2\pi)\log x$, we obtain

$$\sum_{T<n\leq N} \frac{1}{n^s} = \frac{T^{1-s} - N^{1-s}}{s-1} + \int_T^N \psi(x)f'(x)\,dx + O(T^{-\sigma}).$$

Note that

$$\int_T^N \psi(x)f'(x)\,dx = \int_T^N h'(x)\psi(x)e^{2\pi i g(x)}\,dx + 2\pi i \int_T^N h(x)g'(x)\psi(x)e^{2\pi i g(x)}\,dx.$$

It is easily seen that

$$\left|\int_T^N h'(x)\psi(x)e^{2\pi i g(x)}\,dx\right| \leq \frac{1}{2}\int_T^N |h'(x)|\,dx < T^{-\sigma}.$$

Let $h_1(x) := h(x)/x = x^{-\sigma-1}$. Then we have

$$H_1 := \max(|h_1(T)|, |h_1(N)|) + \int_T^N |h_1'(x)|\,dx < 2T^{-\sigma-1}.$$

Since

$$|g'(x)| = \frac{|t|}{2\pi x} \leq \frac{1}{\pi} < 1$$

for all $x \in [T, N]$, it follows by Lemma 2.2 that

$$\left|2\pi i \int_T^N h(x)g'(x)\psi(x)e^{2\pi i g(x)}\,d\right| = |t|\left|\int_T^N h_1(x)\psi(x)e^{2\pi i g(x)}\,d\right| \ll |t|H_1 \ll T^{-\sigma}.$$

We have thus shown that

$$\sum_{T<n\leq N} \frac{1}{n^s} = \frac{T^{1-s} - N^{1-s}}{s-1} + O(T^{-\sigma}). \tag{5}$$

3. An Approximation of $\zeta(s)$

Recall the Riemann zeta-function $\zeta(s)$ is originally defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$. Let N be a positive integer. By partial summation we have

$$\begin{aligned}
\sum_{n=N+1}^{\infty} \frac{1}{n^s} &= -\frac{1}{N^{s-1}} + s\int_N^\infty \frac{\lfloor x \rfloor}{x^{s+1}}\,dx \\
&= -\frac{1}{N^{s-1}} + s\int_N^\infty \frac{1}{x^s}\,dx - \frac{s}{2}\int_N^\infty \frac{1}{x^{s+1}}\,dx - s\int_N^\infty \frac{\psi(x)}{x^{s+1}}\,dx \\
&= \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} - s(s+1)\int_N^\infty \left(\int_N^x \psi(y)\,dy\right)\frac{1}{x^{s+2}}\,dx,
\end{aligned}$$

where $\sigma > -1$. It is easily seen that

$$\int_N^\infty \left(\int_N^x \psi(y)\, dy \right) \frac{1}{x^{s+2}}\, dx \ll \int_N^\infty \frac{1}{x^{\sigma+2}}\, dx = \frac{N^{-\sigma-1}}{\sigma+1},$$

since $\psi(x)$ is periodic of period 1 and

$$\int_0^1 \psi(x)\, dx = 0.$$

Hence

$$\zeta(s) = \sum_{n \le N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} + O\left(\frac{|s(s+1)|N^{-\sigma-1}}{\sigma+1} \right)$$

for $\sigma > -1$ and $N \ge 1$.

Suppose now that $\sigma > 0$ and $|t| \le 2T$ with $1 \le T \le N$. By (5) we have

$$\zeta(s) = \sum_{n \le T} \frac{1}{n^s} + \frac{T^{1-s}}{s-1} - \frac{N^{-s}}{2} + O\left(\frac{|s(s+1)|N^{-\sigma-1}}{\sigma+1} \right) + O(T^{-\sigma}).$$

We obtain the following result [14, Proposition 6.1] by letting $N \to \infty$.

Proposition 3.1. *Let $T \ge 1$ and let $s = \sigma + it \in \mathbb{C}$ with $\sigma > 0$ and $|t| \le 2T$. Then*

$$\zeta(s) = \sum_{n \le T} \frac{1}{n^s} + \frac{T^{1-s}}{s-1} + O(T^{-\sigma}).$$

In particular, Proposition 3.1 implies that $\zeta(1+it) = O(\log t)$ for large t. Taking $s = 1/2 + it$ we obtain the following simple approximation of $\zeta(s)$ on the critical line by its corresponding partial sum.

Corollary 3.2. *Let $T \ge 1$ and let $\delta \in (0, 2)$. Then*

$$\zeta(1/2 + it) = \sum_{n \le T} \frac{1}{n^{1/2+it}} + O(\delta^{-1} T^{-1/2})$$

for all $t \in \mathbb{R}$ with $\delta T \le |t| \le 2T$.

4. The Mean Square of Dirichlet Polynomials

A Dirichlet polynomial $A_N(s)$ of length $N \ge 1$ is a complex-valued function of a complex variable $s = \sigma + it$ of the form

$$A_N(s) := \sum_{n=1}^N \frac{a_n}{n^s}$$

with $a_1, ..., a_n \in \mathbb{C}$. We shall now prove the following result [14, Theorem 13.1] concerning the mean square of $A_N(s)$ over the interval $[0, T]$.

Proposition 4.1. *For $T > 0$ we have*

$$\int_0^T |A_N(s)|^2\, dt = T \sum_{n=1}^N \frac{|a_n|^2}{n^{2\sigma}} + O\left(\left(\sum_{n=1}^N \frac{n^2 |a_n|^2}{n^{2\sigma}} \right)^{\frac{1}{2}} \left(\sum_{n=1}^N \frac{|a_n|^2}{n^{2\sigma}} \right)^{\frac{1}{2}} \right).$$

Proof. Let

$$I(\sigma,T):=\int_0^T |A_N(s)|^2\,dt.$$

Note that

$$I(\sigma,T)=\sum_{m,n=1}^N \frac{a_m\overline{a_n}}{(mn)^\sigma}\int_0^T\left(\frac{n}{m}\right)^{it}dt = T\sum_{n=1}^N\frac{|a_n|^2}{n^{2\sigma}}+\sum_{\substack{1\le m,n\le N\\ m\ne n}}\frac{a_m\overline{a_n}}{(mn)^\sigma}\cdot\frac{(n/m)^{iT}-1}{i\log(n/m)}.$$

By the mean value theorem we have

$$\left|\log\frac{n}{m}\right|\ge\frac{|m-n|}{\max(m,n)}>\frac{|m-n|}{m+n}$$

for every pair (m,n) with $1\le m\ne n\le N$. Thus

$$\sum_{\substack{1\le m,n\le N\\ m\ne n}}\frac{a_m\overline{a_n}}{(mn)^\sigma}\cdot\frac{(n/m)^{iT}-1}{i\log(n/m)}\ll\sum_{\substack{1\le m,n\le N\\ m\ne n}}\frac{|a_m||a_n|}{(mn)^\sigma}\cdot\frac{m+n}{|m-n|}=2\sum_{\substack{1\le m,n\le N\\ m\ne n}}\frac{|a_m||a_n|}{(mn)^\sigma}\cdot\frac{m}{|m-n|}.$$

Now we invoke the following famous inequality of Hilbert:

$$\left|\sum_{1\le m\ne n\le N}\frac{a_m\overline{b_n}}{m-n}\right|\le\pi\left(\sum_{n=1}^N|a_n|^2\right)^{\frac12}\left(\sum_{n=1}^N|b_n|^2\right)^{\frac12},$$

where $a_1,...,a_N,b_1,...,b_N\in\mathbb{C}$. Replacing a_n with $|a_n|/n^{\sigma-1}$ and b_n with $|a_n|/n^\sigma$ in Hilbert's inequality, we obtain

$$\sum_{\substack{1\le m,n\le N\\ m\ne n}}\frac{|a_m||a_n|}{(mn)^\sigma}\cdot\frac{m}{|m-n|}\le\pi\left(\sum_{n=1}^N\frac{n^2|a_n|^2}{n^{2\sigma}}\right)^{\frac12}\left(\sum_{n=1}^N\frac{|a_n|^2}{n^{2\sigma}}\right)^{\frac12}.$$

Hence

$$I(\sigma,T)=T\sum_{n=1}^N\frac{|a_n|^2}{n^{2\sigma}}+O\left(\left(\sum_{n=1}^N\frac{n^2|a_n|^2}{n^{2\sigma}}\right)^{\frac12}\left(\sum_{n=1}^N\frac{|a_n|^2}{n^{2\sigma}}\right)^{\frac12}\right).$$

This completes the proof. □

We obtain the following result [14, Corollary 13.2] as a corollary of Proposition 4.1.

Corollary 4.2. *For $T>0$ we have*

$$\int_0^T|A_N(s)|^2\,dt=(T+O(N))\sum_{n=1}^N\frac{|a_n|^2}{n^{2\sigma}}.$$

Remark 4.1. To make our exposition as self-contained as possible, we give here a short proof of Hilbert's inequality used in the proof of Proposition 4.1. In fact, we shall prove that if $a_1,...,a_n$ and $b_1,...,b_n$ are two sequences of complex numbers, and if $m_1,...,m_n$ are pairwise distinct integers, then

$$\left|\sum_{1\le k\ne l\le n}\frac{a_k\overline{b_l}}{m_k-m_l}\right|\le\pi\left(\sum_{k=1}^n|a_k|^2\right)^{\frac12}\left(\sum_{k=1}^n|b_k|^2\right)^{\frac12}.$$

Note that for any $c \in \mathbb{Z} \setminus \{0\}$,

$$\frac{i}{2\pi} \int_0^{2\pi} (x-\pi) e^{icx}\, dx = \frac{1}{c}.$$

Thus we have

$$\left| \sum_{1 \le k \ne l \le n} \frac{a_k \overline{b_l}}{m_k - m_l} \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} (x-\pi) \sum_{1 \le k \ne l \le n} a_k \overline{b_l}\, e^{i(m_k - m_l)x}\, dx \right|.$$

Since

$$\int_0^{2\pi} (x-\pi)\, dx = 0,$$

it follows that

$$\left| \sum_{1 \le k \ne l \le n} \frac{a_k \overline{b_l}}{m_k - m_l} \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} (x-\pi) \sum_{k=1}^{n} a_k e^{im_k x} \sum_{k=1}^{n} \overline{b_k} e^{-im_k x}\, dx \right|.$$

By Cauchy-Schwarz inequality, the right side is

$$\begin{aligned}
&\le \frac{1}{2\pi} \left(\int_0^{2\pi} (x-\pi)^2 \left| \sum_{k=1}^{n} a_k e^{im_k x} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_0^{2\pi} \left| \sum_{k=1}^{n} b_k e^{im_k x} \right|^2 dx \right)^{\frac{1}{2}} \\
&\le \frac{1}{2} \left(\int_0^{2\pi} \left| \sum_{k=1}^{n} a_k e^{im_k x} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_0^{2\pi} \left| \sum_{k=1}^{n} b_k e^{im_k x} \right|^2 dx \right)^{\frac{1}{2}} \\
&= \pi \left(\sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |b_k|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

This completes the proof. Further generalizations of Hilbert's inequality have been discovered by Montgomery and Vaughan [16].

For a different proof of Proposition 4.1, see [14, Theorem 13.1].

5. The Second Moment of $\zeta(1/2 + it)$

We are now ready to prove the following theorem [6, Theorem 2.41] concerning the mean square of $\zeta(s)$ on the critical line.

Theorem 5.1. *For large $T > 0$ we have*

$$I_1(T) := \int_0^T \left| \zeta\left(\frac{1}{2} + it \right) \right|^2 dt \sim T \log T.$$

Proof. Let $m := \lfloor \log T / \log 2 \rfloor$ and let $T_k := T/2^k$ for $0 \le k \le m+1$. Then $T_k \ge 1$ for all $0 \le k \le m$. For $t \in \mathbb{R}$ with $T_{k+1} < t \le T_k$, we have by Corollary 3.2 that

$$\left| \zeta\left(\frac{1}{2} + it \right) \right|^2 = \left| \sum_{n \le T_k} \frac{1}{n^{1/2+it}} \right|^2 + O(1),$$

since

$$\sum_{n\leq T_k}\frac{1}{n^{1/2+it}}\ll\sum_{n\leq T_k}\frac{1}{n^{1/2}}\ll\sqrt{T_k}.$$

By Proposition 4.1 we have

$$\begin{aligned}\int_{T_{k+1}}^{T_k}\left|\sum_{n\leq T_k}\frac{1}{n^{1/2+it}}\right|^2 dt &= \int_0^{T_{k+1}}\left|\sum_{n\leq T_k}\frac{n^{-iT_{k+1}}}{n^{1/2+it}}\right|^2 dt\\ &= T_{k+1}\sum_{n\leq T_k}\frac{1}{n}+O\left(\left(\sum_{n\leq T_k}n\right)^{\frac{1}{2}}\left(\sum_{n\leq T_k}\frac{1}{n}\right)^{\frac{1}{2}}\right)\\ &= T_{k+1}\log T_k+O(T_k(\log T)^{\frac{1}{2}}).\end{aligned}$$

Hence

$$\int_{T_{m+1}}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt=\sum_{k=0}^{m}T_{k+1}\log T_k+O(T(\log T)^{\frac{1}{2}}).$$

Since $m+1>\log T/\log 2$, we have

$$\sum_{k=0}^{m}T_{k+1}\log T_k=\sum_{k=0}^{m}\frac{T\log T}{2^{k+1}}+O(T)=\left(1-\frac{1}{2^{m+1}}\right)T\log T+O(T)=T\log T+O(T).$$

It follows that

$$\int_{T_{m+1}}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt=T\log T+O(T).$$

But $T_{m+1}=T/2^{m+1}<1$ implies

$$\int_0^{T_{m+1}}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt=O(1).$$

Therefore, we conclude that

$$\int_0^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt=T\log T+O(T(\log T)^{\frac{1}{2}}).$$

This completes the proof. □

Remark 5.1. More precise asymptotics for $I_1(T)$ are known. For instance, one can show [20, Theorem 7.4] that

$$I_1(T)=T\log T+(2\gamma-1-\log 2\pi)T+O(T^{1/2+\epsilon})$$

for any given $\epsilon>0$, where γ is Euler's constant. Balasubramanian [2] further reduced the exponent $1/2$ in the error term down to $1/3$.

6. Titchmarsh's Approach

In 1927 Titchmarsh [19] gave a proof of Theorem 5.1 using tools from Fourier analysis without much reference to deep properties of $\zeta(s)$. Here we describe briefly the main ideas behind his proof. Recall that for any function $f(x) \in L^1(\mathbb{R})$, the Fourier transform $F(\xi)$ of f is defined by

$$F(\xi) := \int_{-\infty}^{+\infty} f(x)e^{-2\pi i\xi x}\, dx.$$

The well-known Plancherel's formula states that if $f(x), g(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with Fourier transforms $F(\xi)$ and $G(\xi)$, respectively, then

$$\int_{-\infty}^{+\infty} f(x)\overline{g(x)}\, dx = \int_{-\infty}^{+\infty} F(\xi)\overline{G(\xi)}\, d\xi. \tag{6}$$

In particular, one has

$$\int_{-\infty}^{+\infty} |f(x)|^2\, dx = \int_{-\infty}^{+\infty} |F(\xi)|^2\, d\xi. \tag{7}$$

Let $\eta \in \mathbb{R}$ be a real variable independent of x and ξ. Since the Fourier transform of $f(x)e^{2\pi i\eta x}$ is $F(\xi - \eta)$, it follows by (6) that

$$\int_{-\infty}^{+\infty} |f(x)|^2 e^{-2\pi i\eta x}\, dx = \int_{-\infty}^{+\infty} F(\xi)\overline{F(\xi-\eta)}\, d\xi.$$

This means that the Fourier transform of $|f(x)|^2$ is

$$\int_{-\infty}^{+\infty} F(\xi)\overline{F(\xi-\eta)}\, d\xi = \int_{-\infty}^{+\infty} F(\xi+\eta)\overline{F(\xi)}\, d\xi.$$

By (7) we have

$$\int_{-\infty}^{+\infty} |f(x)|^4\, dx = \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} F(\xi+\eta)\overline{F(\xi)}\, d\xi \right|^2 d\eta. \tag{8}$$

Now we consider the Riemann zeta-function $\zeta(s)$. We begin with the Cahen-Mellin integral

$$e^{-z} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s}\, ds,$$

valid for $z \in \mathbb{C}$ with $\Re(z) > 0$ and $c > 0$, where

$$\Gamma(s) := \int_0^{\infty} x^{s-1}e^{-x}\, dx.$$

For $c > 1$ we have

$$\frac{1}{e^z - 1} = \sum_{n=1}^{\infty} e^{-nz} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s) z^{-s}\, ds,$$

where the interchange of summation and integration is easily justified using Stirling's formula. Moving the line of integration to $\Re(s) = 1/2$ and taking into account the simple pole of the intergrand at $s = 1$ with residue $1/z$, we obtain

$$\frac{1}{e^z-1} - \frac{1}{z} = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \Gamma(s)\zeta(s)z^{-s}\, ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Gamma\left(\frac{1}{2}+it\right)\zeta\left(\frac{1}{2}+it\right) z^{-1/2-it}\, dt.$$

Taking $z = ie^{2\pi\xi - i\delta}$ with $0 < \delta < 1$, we have

$$\frac{1}{\exp(ie^{2\pi\xi - i\delta}) - 1} - \frac{1}{ie^{2\pi\xi - i\delta}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Gamma\left(\frac{1}{2} + it\right)\zeta\left(\frac{1}{2} + it\right) e^{-\pi\xi - i(\pi/2-\delta)(1/2+it)} e^{-2\pi i\xi t}\, dt.$$

This shows that the Fourier transform of

$$\frac{1}{2\pi}\Gamma\left(\frac{1}{2} + it\right)\zeta\left(\frac{1}{2} + it\right) e^{-i(\pi/2-\delta)(1/2+it)}$$

is

$$e^{\pi\xi}\left(\frac{1}{\exp(ie^{2\pi\xi - i\delta}) - 1} - \frac{1}{ie^{2\pi\xi - i\delta}}\right).$$

Applying (7) we obtain $L(\delta) = R(\delta)$, where

$$L(\delta) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} \left|\Gamma\left(\frac{1}{2} + it\right)\right|^2 \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 e^{(\pi-2\delta)t}\, dt,$$

$$R(\delta) = \int_{-\infty}^{+\infty} e^{2\pi\xi}\left|\frac{1}{\exp(ie^{2\pi\xi - i\delta}) - 1} - \frac{1}{ie^{2\pi\xi - i\delta}}\right|^2 d\xi.$$

By Euler's reflection formula we have

$$\left|\Gamma\left(\frac{1}{2} + it\right)\right|^2 = \Gamma\left(\frac{1}{2} + it\right)\Gamma\left(\frac{1}{2} - it\right) = \frac{\pi}{\sin\pi(1/2+it)} = \frac{\pi}{\cosh\pi t}.$$

Since

$$\frac{1}{\cosh\pi t} = \frac{2}{e^{\pi t} + e^{-\pi t}} = \frac{2e^{-\pi|t|}}{1 + e^{-2\pi|t|}} = 2e^{-\pi|t|}\left(1 - \frac{e^{-2\pi|t|}}{1 + e^{-2\pi|t|}}\right) = 2e^{-\pi|t|}(1 + O(e^{-2\pi|t|})),$$

it follows that

$$\left|\Gamma\left(\frac{1}{2} + it\right)\right|^2 = 2\pi e^{-\pi|t|} + O(e^{-3\pi|t|}).$$

Thus we have

$$L(\delta) = \frac{1}{2\pi}\int_0^\infty \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 e^{-2\delta t}\, dt + O\left(\int_0^\infty \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 e^{-2(\pi-\delta)t}\, dt\right).$$

Note that

$$\zeta(s) = s\int_1^\infty \frac{\lfloor x\rfloor}{x^{s+1}}\, dx = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\, dx$$

for $s = \sigma + it \in \mathbb{C}$ with $\sigma > 0$. Taking $\sigma = 1/2$ we see that $\zeta(1/2 + it) = O((1 + |t|))$. Hence

$$L(\delta) = \frac{1}{2\pi}\int_0^\infty \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 e^{-2\delta t}\, dt + O(1).$$

A Tauberian result [6, Lemma 2.413] implies that Theorem 5.1 is equivalent to the following theorem [19, Theorem I].

Theorem 6.1. *As $\delta \to 0^+$ we have*

$$\int_0^\infty \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 e^{-\delta t}\, dt \sim \frac{1}{\delta}\log\frac{1}{\delta}.$$

Consequently, our original task of proving Theorem 5.1 is reduced to the estimation of $R(\delta)$ with the goal of showing that

$$R(\delta) \sim \frac{1}{4\pi\delta}\log\frac{1}{\delta} \tag{9}$$

as $\delta \to 0^+$. Performing the substitution $\eta = e^{2\pi\xi}$ we get

$$\begin{aligned} R(\delta) &= \frac{1}{2\pi}\int_0^\infty \left|\frac{1}{\exp(i\eta e^{-i\delta})-1} - \frac{1}{i\eta e^{-i\delta}}\right|^2 d\eta \\ &= \frac{1}{2\pi}\int_\pi^\infty \left|\frac{1}{\exp(i\eta e^{-i\delta})-1} - \frac{1}{i\eta e^{-i\delta}}\right|^2 d\eta + O(1), \end{aligned}$$

since $1/(e^z-1) - 1/z$ is analytic in $|z| < 2\pi$. Expand out the integrand and observe that the main contribution comes from the term

$$\frac{1}{2\pi}\int_\pi^\infty \left|\frac{1}{\exp(i\eta e^{-i\delta})-1}\right|^2 d\eta = \frac{1}{2\pi}\int_\pi^\infty \frac{d\eta}{(1-\exp(i\eta e^{-i\delta}))(1-\exp(-i\eta e^{i\delta}))}.$$

Note that $|\exp(-i\eta e^{-i\delta})| = |\exp(i\eta e^{i\delta})| = e^{-\eta\sin\delta} < 1$ when $0 < \delta < 1$. Using the power series expansion $z/(z-1) = \sum_{n=1}^\infty z^n$ valid for $|z| < 1$ we get

$$\frac{1}{(1-\exp(i\eta e^{-i\delta}))(1-\exp(-i\eta e^{i\delta}))} = \sum_{m,n=1}^\infty \exp\left(-im\eta e^{-i\delta} + in\eta e^{i\delta}\right).$$

The contribution from the diagonal terms gives

$$\frac{1}{2\pi}\int_\pi^\infty \sum_{n=1}^\infty e^{-2n\eta\sin\delta}\, d\eta = \frac{1}{2\pi}\int_\pi^\infty \frac{e^{-2\eta\sin\delta}}{1-e^{-2\eta\sin\delta}}\, d\eta \sim \frac{1}{4\pi\delta}\int_\pi^\infty \frac{e^{-2\eta\sin\delta}}{\eta}\, d\eta.$$

By integration by parts we obtain

$$\int_\pi^\infty \frac{e^{-2\eta\sin\delta}}{\eta}\, d\eta = \int_{2\pi\sin\delta}^\infty \frac{e^{-\eta}}{\eta}\, d\eta = -e^{-2\pi\sin\delta}\log(2\pi\sin\delta) + \int_{2\pi\sin\delta}^\infty e^{-\eta}\log\eta\, d\eta.$$

Since the improper integral

$$\int_0^\infty e^{-\eta}\log\eta\, d\eta = \int_0^1 e^{-\eta}\log\eta\, d\eta + \int_1^\infty e^{-\eta}\log\eta\, d\eta$$

is absolutely convergent, we have

$$\int_\pi^\infty \frac{e^{-2\eta\sin\delta}}{\eta}\, d\eta = -e^{-2\pi\sin\delta}\log\sin\delta + O(1) \sim \log\frac{1}{\delta}.$$

This leads to

$$\frac{1}{2\pi}\int_\pi^\infty \sum_{n=1}^\infty e^{-2n\eta\sin\delta}\, d\eta \sim \frac{1}{4\pi\delta}\log\frac{1}{\delta}.$$

On the other hand, it is not hard to show that the contribution from the off-diagonal terms

$$\frac{1}{2\pi}\sum_{\substack{m,n=1\\ m\neq n}}^\infty \int_\pi^\infty \exp\left(-im\eta e^{-i\delta} + in\eta e^{i\delta}\right) d\eta = \frac{1}{2\pi}\sum_{\substack{m,n=1\\ m\neq n}}^\infty \frac{\exp(-(m+n)\pi\sin\delta - i(m-n)\pi\cos\delta)}{(m+n)\pi\sin\delta + i(m-n)\pi\cos\delta}$$

is $O(\delta^{-1})$. This proves (9). The reader is referred to [19, Theorem I] for further details.

7. Concluding Remarks

More generally, one can study the $2k$-th moment of $\zeta(s)$ on the critical line defined by

$$I_k(T) := \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt$$

for any $k > 0$. The study of $I_k(T)$ for large T plays a significant role in the theory of the Riemann zeta-function. For instance, the famous Lindelöf hypothesis states that for every $\epsilon > 0$ we have $\zeta(1/2+it) = O(t^\epsilon)$ for large $t > 0$. This hypothesis, if true, would tell us about the location of the nontrivial zeros of ζ. Indeed, Backlund [1] showed that the Lindelöf hypothesis is equivalent to the statement that for every $\epsilon > 0$ the number of zeros $\rho = \sigma + it$ of ζ with $\sigma \geq 1/2+\epsilon$ and $T \leq t \leq T+1$ is $o(\log T)$ as $T \to \infty$. This is of course weaker than the Riemann hypothesis which states that all the zeros $\rho = \sigma+it$ of ζ with $\sigma \geq 0$ must lie on the critical line $\sigma = 1/2$. In fact, Littlewood [15] proved that the Riemann hypothesis implies that $\zeta(1/2+it) = O(t^{C/\log\log t})$ for large $t > 0$, where C is a positive constant, and it has been shown by Chandee and Soundararajan [3] that one can take arbitrary $C > \log\sqrt{2}$. Such information would have important implications on the error term in the approximation of the prime counting function $\pi(x)$ by the logarithmic integral $\mathrm{li}(x)$ defined by

$$\mathrm{li}(x) := \int_2^x \frac{dt}{\log t},$$

as well as on gaps between consecutive primes. On the other hand, Hardy and Littlewood showed that the Lindelöf hypothesis is equivalent to the statement that for every $\epsilon > 0$ and every positive integer $k \geq 1$ we have $I_k(T) = O(T^{1+\epsilon})$. These equivalences make the study of higher moments of ζ on the critical line especially meaningful. Currently the Lindelöf hypothesis is still open, though it is known that $\zeta(1/2+it) = O(t^{\frac{1}{4}})$ for large $t > 0$ (called the "convexity bound") and various results of the form $\zeta(1/2+it) = O(t^\alpha(\log t)^\beta)$ with $0 < \alpha < 1/4$ have been obtained. See [20, Chapter V] for more details.

For $k = 2$, Hardy and Littlewood [7] showed by using the approximate functional equation for $\zeta(s)$ that $I_2(T) = O(T(\log T)^4)$. Using the approximate functional equation for $\zeta(s)^2$, Ingham [12] proved the following asymptotic for $I_2(T)$:

$$I_2(T) = \frac{1}{2\pi^2}T(\log T)^4 + O(T(\log T)^3).$$

A proof of this result using (8) was found later by Titchmarsh [19]. Both proofs make use of the following formula of Ramanujan [17]:

$$\sum_{n\leq x} d(n)^2 = \frac{1}{\pi^2}x(\log x)^3 + O(x(\log x)^2), \tag{10}$$

where $d(n)$ counts the number of positive divisors of n. The interested reader can find a proof of this formula in the appendix. One important feature of Titchmarsh's method is that it can also be used to determine in a similar vein the asymptotics for the second and fourth moments of

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s}$$

on $\sigma = 1/2$. Unfortunately, when $k \neq 1, 2$ no asymptotics for $I_k(T)$ are known. It is conjectured that for any $k > 0$ one has $I_k(T) \sim c_k T(\log T)^{k^2}$ for some constant $c_k > 0$. It is also conjectured that if $k \geq 1$ is a positive integer, then

$$c_k = \frac{g_k a_k}{\Gamma(k^2+1)},$$

where g_k is a positive integer and

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{(k-1)^2} \left(\sum_{l=0}^{k-1} \binom{k-1}{l}^2 p^{-l}\right).$$

The infinite product on the right side is easily seen to be convergent. Models from random matrix theory seem to suggest that

$$g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

For example, for $k = 2$ we have

$$a_2 = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

and $g_2 = 2$. Thus $c_2 = 1/(2\pi^2)$, which matches the coefficient of the main term in Ingham's asymptotic for $I_2(T)$. It can be shown [4] that g_k defined this way is indeed a positive integer for every $k \geq 1$. In fact, g_k can be interpreted as the number of standard Young tableaux of shape $k \times k$ (see [5]). Though a proof or disproof of the conjectured asymptotic above seems elusive, much progress has been made toward sharp upper and lower bounds for $I_k(T)$ with k in certain ranges. For instance, it has been shown in [10] that if $0 \leq k \leq 2$, then $I_k(T) \ll T(\log T)^{k^2}$ for $T \geq e$. Assuming the Riemann hypothesis, Harper [9] proved that this estimate holds for every $k \geq 0$, refining an earlier result of Soundararajan [18]. As for sharp lower bounds, Heap and Soundararajan [11] recently showed that for any large T and any fixed $\delta > 0$, we have $I_k(T) \geq C_k T(\log T)^{k^2}$ uniformly for $(\log T)^{-\frac{1}{2}} \leq k \leq (\log T)^{\frac{1}{2}-\delta}$, where $C_k > 0$ is some constant. Thus for every $k \geq 0$, one has $I_k(T) \gg T(\log T)^{k^2}$ for $T \geq e$.

8. Appendix: Proof of Ramanujan's Formula

We give a self-contained proof of Ramanujan's formula (10). In the proof we shall see the intimate connection between the Riemann zeta-function and various arithmetic functions. The starting point is the following identity involving the generating function of $d(n)^2$ [8, Theorem 304]:

$$\sum_{n=1}^{\infty} \frac{d(n)^2}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)} \tag{11}$$

for $s \in \mathbb{C}$ with $\Re(s) > 1$. The proof of this identity is easy. Using the Euler product for $\zeta(s)$ we obtain

$$\frac{\zeta(s)^4}{\zeta(2s)} = \prod_p \frac{1 - p^{-2s}}{(1-p^{-s})^4} = \prod_p \frac{1 + p^{-s}}{(1-p^{-s})^3}.$$

Note that

$$\begin{aligned}\frac{1+p^{-s}}{(1-p^{-s})^3} &= (1+p^{-s})\sum_{k=0}^{\infty}(-1)^k\binom{-3}{k}p^{-ks}\\ &= (1+p^{-s})\sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}p^{-ks}\\ &= \sum_{k=0}^{\infty}(k+1)^2p^{-ks}\\ &= \sum_{k=0}^{\infty}d(p^k)^2p^{-ks}.\end{aligned}$$

Since $d(n)$ is multiplicative, we have

$$\frac{\zeta(s)^4}{\zeta(2s)} = \prod_p\sum_{k=0}^{\infty}d(p^k)^2p^{-ks} = \sum_{n=1}^{\infty}\frac{d(n)^2}{n^s}$$

as desired. Next, we derive the Dirichlet series expansion of $\zeta(s)^4/\zeta(2s)$ in a different way. Let $d_k(n)$ denote the number of representations of n as the product of k positive divisors of n. Formally, we have

$$d_k(n) := \#\{(d_1, ..., d_k) \in \mathbb{N}^k \colon n = d_1\cdots d_k\}.$$

Thus $d_1(n) = 1$ and $d_2(n) = d(n)$. By Dirichlet convolution we have

$$\frac{\zeta(s)^4}{\zeta(2s)} = \left(\sum_{n=1}^{\infty}\frac{d_4(n)}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{2s}}\right) = \sum_{n=1}^{\infty}\left(\sum_{uv^2=n}d_4(u)\mu(v)\right)\frac{1}{n^s}$$

when $\Re(s) > 1$, where $\mu(n)$ is the Möbius function. Comparing this with (11) we obtain

$$d(n)^2 = \sum_{uv^2=n}d_4(u)\mu(v). \tag{12}$$

It is now clear that we need an asymptotic formula for the summatory function

$$D_k(x) := \sum_{n\le x}d_k(n),$$

where $k \ge 2$ is a positive integer. It turns out that we also need to estimate

$$E_k(x) := \sum_{n\le x}\frac{d_k(n)}{n}.$$

Note that

$$D_2(x) = \sum_{n\le x}d(n) = \sum_{h\le x}\left\lfloor\frac{x}{h}\right\rfloor = x\sum_{h\le x}\frac{1}{h} + O(x) = x\log x + O(x). \tag{13}$$

More precise estimates for $D_2(x)$ are known, but this crude one will suffice for our purpose. By partial summation we obtain

$$E_2(x) = \frac{D_2(x)}{x} + \int_1^x\frac{D_2(t)}{t^2}\,dt = \frac{1}{2}(\log x)^2 + O(\log x). \tag{14}$$

To deal with $D_k(x)$ and $E_k(x)$ for $k \geq 3$, we make use of the following recursive formula:

$$d_k(n) = \sum_{h|n} d_{k-1}(h),$$

where $k \geq 3$. It follows that

$$D_k(x) = \sum_{h \leq x} \left\lfloor \frac{x}{h} \right\rfloor d_{k-1}(h) = xE_{k-1}(x) + O(D_{k-1}(x)),$$

$$E_k(x) = \frac{D_k(x)}{x} + \int_1^x \frac{D_k(t)}{t^2}\, dt,$$

where $k \geq 3$. Combining these formulas with (13) and (14) we obtain by induction that

$$D_k(x) = \frac{1}{(k-1)!} x(\log x)^{k-1} + O(x(\log x)^{k-2}),$$

$$E_k(x) = \frac{1}{k!}(\log x)^k + O((\log x)^{k-1}),$$

where $k \geq 2$. Now (10) can be derived from (11) by means of the above formula for $D_k(x)$. Indeed, we have

$$\sum_{n \leq x} d(n)^2 = \sum_{v \leq \sqrt{x}} \mu(v) D_4\left(\frac{x}{v^2}\right) = \frac{1}{6} x \sum_{v \leq \sqrt{x}} \frac{\mu(v)}{v^2}\left(\log \frac{x}{v^2}\right)^3 + O(x(\log x)^2).$$

Note that

$$\sum_{v \leq \sqrt{x}} \frac{\mu(v)}{v^2}\left(\log \frac{x}{v^2}\right)^3 = (\log x)^3 \sum_{v \leq \sqrt{x}} \frac{\mu(v)}{v^2} + O((\log x)^2).$$

Since

$$\sum_{v \leq \sqrt{x}} \frac{\mu(v)}{v^2} = \frac{1}{\zeta(2)} + O\left(\sum_{v > \sqrt{x}} \frac{1}{v^2}\right) = \frac{6}{\pi^2} + O(x^{-1/2}),$$

we have

$$\sum_{v \leq \sqrt{x}} \frac{\mu(v)}{v^2}\left(\log \frac{x}{v^2}\right)^3 = \frac{6}{\pi^2}(\log x)^3 + O((\log x)^2).$$

Hence

$$\sum_{n \leq x} d(n)^2 = \frac{1}{\pi^2} x(\log x)^3 + O(x(\log x)^2).$$

This completes the proof of (10).

Dirichlet discovered the following refinement of (13) [8, Theorem 320]:

$$D_2(x) = \sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

The technique employed in his proof is now known as the Dirichlet hyperbola method, for the reason that geometrically, the summation is arranged to be over the lattice points $(a, b) \in \mathbb{N}^2$ under the hyperbola $ab = x$. It is often very useful for estimating the summatory function of the Dirichlet convolution of two arithmetic functions. The interested reader is referred to [13, Chapters 13 & 14] for more precise asymptotic formulas for $D_k(x)$ and the sum $\sum_{n\le x} d(n)^2$, where the deep theory of the Riemann zeta-function is exploited.

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